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THE CHROMOSOMES OF LATHYRUS TUBEROSUS

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Considerable cytological and genetical work has been done on the genus Lathyrus. The present paper deals particularly with a perennial species, *Lathyrus tuberosus* L., upon which the writer finds no previous report.

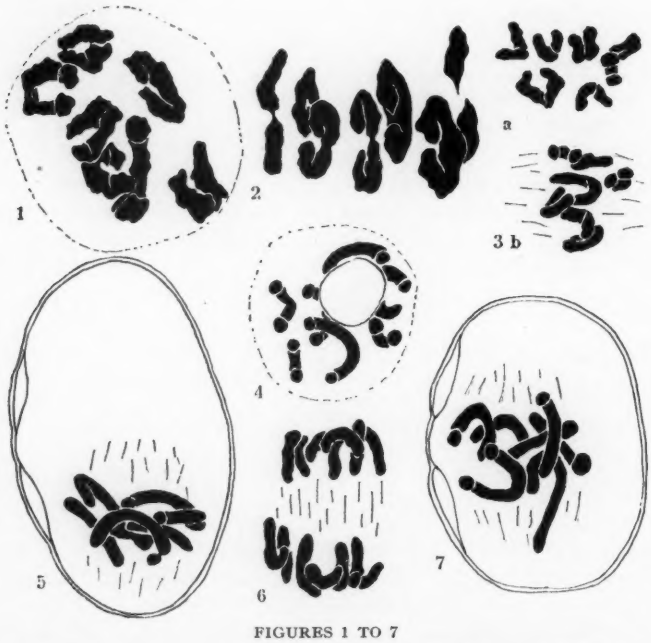
Chromosome counts have been reported in the genus Lathyrus as follows:

	n	$2n$	
Lathyrus Aphaca,	7		Corti 1930[1]
L. latifolius	7	14	Winge 1919[2]
L. maritimus	7		Kawakami 1930[3]
L. odoratus	7	14	Winge 1919[3]
" "	7		Latter 1926[4]
" "	7		Punnett 1927[5]
" "	7	14	Maeda 1928,[6] 1930[7]
" "	7		Kawakami 1930[3]
L. vernus		14	Sakamura 1920[8]

Material of *L. tuberosus* has been obtained from plants growing in an orchard on the campus of the University of Wisconsin and has been prepared by the aceto-carmine smear method. Fixations have also been made with Carnoy's and Flemming's medium fluids.

A study has been made of the chromosomes in stages of diakinesis and metaphase of the heterotypic division and at the equatorial plate of the homoeotypic division (Figs. 1–3). In all cases the haploid chromosome number is found to be seven, thus agreeing with the number in other species already reported. The chromosomes are unusually large in comparison with those of many other genera of plants. A decided variation in size and in shape of chromosomes is noted. During diakinesis and the metaphases of the heterotypic division there is some indication that at least five of the pairs have the form of rings, complete or nearly complete (Fig. 2). The members of one pair appear to separate slightly earlier than the others (Fig. 2). A similarity is noted in size and in shape between the chromosomes of *L. tuberosus* and those of *L. odoratus* figured

by Winge,[2] Latter[4] and Maeda.[6,7] There is some indication of the spiral structure of chromosomes described by Maeda[6] for *L. odoratus*, but further study is necessary to determine the nature of the internal differentiation evident at most ages in the material of *L. tuberosus* examined. The microspores are formed simultaneously by the furrowing of the plasma membrane of the mother cell.



FIGURES 1 TO 7

Figures 1–3. Chromosomes of *Lathyrus tuberosus* during microsporogenesis. Figure 1, late diakinesis. Figure 2, metaphase, heterotypic division. Figure 3, *a*, *b*, metaphase, homoeotypic division, in one microsporocyte. Figures 4–6. Chromosomes from dividing microspore of *L. tuberosus*. Figure 4, prophase. Figure 5, equatorial plate. Figure 6, telophase. Figure 7. *L. latifolius*. Equatorial plate, dividing microspore. ×, about 1600.

Interesting stages in the division of the microspore nucleus of *L. tuberosus* have been found. Relatively few drawings of chromosomes from microgametophytes of Angiosperms have been shown in literature. It has been possible to determine the haploid number from counts during prophases, metaphases and telophases in the division of the microspore nucleus (Figs. 4–6). In all cases examined the haploid number in *L.*

tuberosus is definitely seven. There is some indication of a deep constriction near one end of one of the longer chromosomes (Fig. 5).

Prophase, metaphase, and telophase division figures of the microspore nucleus of *L. latifolius* have also been observed. The microspores of this species (Fig. 7) are slightly smaller, and the chromosomes during division occupy a greater proportion of the cell than do those of *L. tuberosus* (Fig. 5) On the basis of these figures, the writer is able to confirm Winge's[2] count of seven for *L. latifolius* as the haploid number of chromosomes for this species.

[1] Corti, R., *N. Giorn. Bot. Ital.*, **37**, 278–279 (1930).
[2] Winge, Ö., *J. Geneti.*, **8**, 133–138 (1919).
[3] Kawakami, J., *Bot. Mag. Tokyo*, **44**, 319–329 (1930).
[4] Latter, Joan, *Annals. Bot.*, **40**, 277–313 (1926).
[5] Punnett, R. C., *Proc. Roy. Soc. London*, **B102**, 236–238 (1927).
[6] Maeda, T., *Bot. Mag. Tokyo*, **42**, 191–195 (1928).
[7] Maeda, T., *Mem. Coll. Sci. Kyoto Imp. Univ.*, **B5**, 89–123 (1930).
[8] Sakamura, T., *J. Coll. Sci. Tokyo*, **39**, 1–221 (1920).

EVIDENCE FOR MULTIPLE SEX FACTORS IN THE X-CHROMOSOME OF DROSOPHILA MELANOGASTER

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The data presented here permit a decision between two alternative hypotheses regarding sex determination in Drosophila melanogaster. Bridges[1,2,3] has shown that the sex of an individual in this form is determined by a relation between the number of the X-chromosomes and that of the autosomes. From his studies of a variety of sex types (table 1), he has concluded that the X-chromosome turns the course of development toward femaleness, the autosomes toward maleness.

Further material for the analysis of this problem comes from the study of the effects of fragments of the X-chromosome in diploids (L. V. Morgan,[4] Painter and Muller,[5] Dobzhansky, unpublished). Here it appears that the addition of certain of the duplications for parts of the X to the chromosome complement of a diploid male or female produces little or no effect on the sex type. Yet it should be noted that, while this is true for the shorter fragments, when longer fragments are involved, males become sterile and otherwise abnormal, and females become more like superfemales.

Two hypotheses have been proposed to account for the data. As a result of his work on triploid intersexes, Bridges advanced the idea that,

like any other group of characters, those concerned with sex are determined by the action of many genes. These are so distributed through the chromosomes that the X contains a preponderance of female-determining genes, the autosomes relatively more of those that are male-determining. From this point of view, the course of development depends on the ratio between the numbers of the genes of different types.

This theory of "genic balance" is sufficient to account for the data. It is not, however, necessary. An alternative hypothesis has been offered by Goldschmidt, on the basis of his work with Lymantria[6]—a system in which sex is determined by the relation between a single female factor (in the X in Drosophila) and a single male factor (in the autosomes). With the additional assumptions of modifying factors and of variations in the "strength" of the primary genes, such a system can be made to account for the previously published data on intersexes. A similar conclusion has been indicated by Painter and Muller[5] and Muller,[7] from the

TABLE 1

RELATION OF SEX TO CHROMOSOMES IN DROSOPHILA (AFTER BRIDGES, LOC. CIT.)

SEX	X-CHROMOSOMES	SETS OF AUTOSOMES	SEX INDEX
Superfemale	3	2	1.5
Female	1	1	1.0
	2	2	1.0
	3	3	1.0
	4	4	1.0
Intersex	2	3	0.67
Male	1	2	0.50
Supermale	1	3	0.33

fertility of males carrying a duplicating fragment of the X-chromosome—specifically, that the "sex differentiators" were not present in such a fragment.

This does not, however, seem evident to us, since it is clear that the development of the sex characters may involve threshold reactions (Goldschmidt, Dobzhansky, loc. cit.). Therefore, we expected that, although in the diploid duplications produced no change, the study of their effects upon intersexes might provide more conclusive data. The intersexes are quite sensitive to both genetic and environmental influences, presenting a series from extreme male to extreme female type. These changes may be brought about by genes which show no appreciable effect on the normal males and females (Dobzhansky[8,9]). Likewise, temperature is effective in changing the grade of intersexual development, whereas in the normal sexes, no effect is obvious. On this basis, we should expect the intersexes to provide a more sensitive measure of the presence of sex factors in a given section of chromosome than the males or females.

The results of our experiments have justified this point of view. We find that in every case, the intersexes containing an additional frag-

ment of the X-chromosome are more female type than those without the fragment. This affords a definite confirmation of the multiple factor hypothesis suggested by Bridges.

Our studies concern six duplications of different lengths, whose characteristics are shown in table 2.

TABLE 2

THE LOCI IN THE X-CHROMOSOME INCLUDED (+) AND DELETED (−) IN THE DUPLICATIONS

	Loci															
DUPLICATION	y	sc	svr	kz	br	pn	w	cv	v	g	r	f	B	fu	cr	bb
112	+	+	+	−	−	−	−	−	−	−	−	−	−	−	−	−
107	+	+	+	−	−	−	−	−	−	−	−	−	−	−	−	+
118	+	+	+	−	−	−	−	−	−	−	−	−	−	−	−	+
134	+	+	+	+	+	−	−	−	−	−	−	−	−	−	−	−
126	−	−	−	−	−	−	−	−	−	−	+	+	+	−	−	−
L. V. M.	+	+	+	+	+	+	−	−	−	−	−	−	−	+	+	+

Individuals carrying the duplications (except in the case of Dup. L. V. M.) are very nearly normal in appearance, and both sexes are fertile. There are minor differences from the wild type—occipital bristles are present, other bristles are occasionally duplicated and there are slight abnormalities in the wing shape. Duplication L. V. M. flies are more strikingly different from normal, and the males are sterile. The duplications 112, 107, 118 and L. V. M. belong to the class of "deletions" (Painter and Muller).[4] Cytologically, 112 is the shortest, 107 next longer, then 118, and L. V. M. is the longest of all. Duplication 126 is a very short section of the middle of the X, attached to chromosome III. Finally, Duplication 134, which has not yet been studied cytologically, belongs to the peculiar "eversporting" type, like that of Patterson and Painter.[10]

For those duplications which produce fertile males, the technique of procuring the intersexes is simple. Stocks of triploids were obtained homozygous for a gene included in and suppressed by the duplication (e.g., yellow for 112, 107, 118 and 134, forked for 126). To these triploids were mated males carrying the duplication and containing in the normal X the mutant gene in question. This procedure allowed the comparison, in the progeny, of two types of intersexes out of the same culture—the intersexes carrying the duplication which have the mutant-gene suppressed, and the intersexes free from the duplication which manifest the characters of the mutant gene.

Males carrying the L. V. M. Duplication are sterile. For this type, another technique was attempted, the production of the duplication-carrying triploids by means of the third-chromosome recessive gene discovered by Gowen. This attempt proved unsuccessful, due to the high

sterility of the stocks; but in an unrelated experiment involving the duplication, a number of triploids appeared spontaneously. A stock of such triploids could unfortunately not be established, due to the mode of disjunction of the X's and the duplication. Nevertheless, enough intersexes carrying the duplication were produced to make the effect of this, the longest duplication involved in the experiment, fully apparent.

Intersexes which appeared in the cultures were classified according to their sexual type. Six classes of intersexes were distinguished—class I is the extreme male-type, classes V and VI are the extreme female type (class VI has no sex-combs). These classes are based on the characters of the external genitalia, which are, however, correlated with the internal structures as well as with the secondary sexual characters (Dobzhansky, loc. cit.). These classes are, of course, arbitrary, and we are unable at present to ascribe to the differences between them a functional significance expressed in quantitative terms.

The results obtained are shown in tables 3 and 4.

TABLE 3

THE INFLUENCE OF THE DUPLICATIONS ON THE TYPE OF THE INTERSEXES

CLASSES OF THE INTERSEXES	DUPLICATION-CARRYING							CONTROL						
	I	II	III	IV	V	VI	N	I	II	III	IV	V	VI	N
Duplication														
126	8	26	17	21	1	—	73	5	9	7	2	1	—	24
112	—	6	10	100	44	1	161	40	50	42	53	1	—	186
107	—	—	5	129	85	9	228	58	116	74	58	—	—	306
118	—	—	10	91	87	15	203	66	59	28	16	—	—	169
134	—	—	2	36	47	11	96	61	188	91	91	2	—	433
L. V. M.	—	—	—	—	—	11	11	—	4	7	—	—	—	11

TABLE 4

THE MEAN TYPE OF THE INTERSEXES CARRYING THE DUPLICATION AND FREE FROM IT

DUPLICATION	CARRYING THE DUPLICATION	CONTROL	DIFFERENCE
126	2.74 ± 0.12	2.38 ± 0.21	0.36 ± 0.24
112	4.15 ± 0.055	2.59 ± 0.08	1.56 ± 0.10
107	4.43 ± 0.04	2.43 ± 0.06	2.00 ± 0.07
118	4.53 ± 0.05	1.97 ± 0.075	2.56 ± 0.09
134	4.70 ± 0.07	2.50 ± 0.06	2.20 ± 0.09
L. V. M.	6.00	2.6	3.4

It is obvious from tables 3 and 4 that the presence of the duplications results in a shift in the type of the intersexes toward the female. Furthermore, it is apparent that approximately the longer the duplication, the greater the shift in the type of intersex, from a very slight shift in the shortest, Dup. 126, until in the longest duplication (L. V. M.) all the intersexes are of the extreme female type. This can only be interpreted to

mean that in all portions of the X thus far studied, female-determining factors are present, and the degree of femaleness is proportional to the number of these factors.

This demonstration is completed by the fertility of one of the extreme female type intersexes carrying duplication L. V. M. Out of five such intersexes tested, this one proved fertile. She produced thirteen diploid offspring, tests of which showed that she contained two normal X's and the duplication; and one ordinary male-type intersex, proving her triploid for the autosomes. In order to strengthen this, measurements were made of the cell size in her wings, following the technique of Dobzhansky.[11] The results were undoubtedly triploid or intersex, removing the possibility that she might be a diploid duplication female in which there had been another occurrence of triploidy. This gives an almost diagrammatic form to the evidence: in the diploid, this duplication is sufficient only to give a sterile male; but added to the chromosome complement of an intersex, it produces a fertile female.

As we have already pointed out, these data fully uphold Bridges' hypothesis of genic balance. They are of especial value in this connection, since they cannot be construed as "mere" modifying factors—an arbitrary distinction at best. They show the same quantitative relations that have been defined as characteristic of the sex factors proper.

This method of analysis may be used further to map out the distribution of the sex factors in the X and in the autosomes, and even to evaluate their relative effectiveness. That there should be such differences in effectiveness is quite probable, and it is possible that some portions of the X should be male-determining and of the autosomes female-determining. It is now obvious, however, that sex in Drosophila is determined by the interaction of a number of genes, rather than by the relation between a single male and female factor.

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[2] Bridges, C. B., "Aberrations in Chromosome Materials," *Eugenics, Genetics and the Family*, **1**, 76–81 (1923).

[3] Bridges, C. B., "Sex in Relation to Chromosomes and Genes," *Amer. Natur.*, **59**, 127–137.

[4] Morgan, L. V., in Morgan, T. H., A. H. Sturtevant and C. B. Bridges, "The Constitution of the Germinal Material in Relation to Heredity," *Carnegie Inst. Year Book No. 27*, 330–335 (1928).

[5] Painter, T. S., and H. J. Muller, "The Parallel Cytology and Genetics of Induced Translocations and Deletions in Drosophila," *J. Hered.*, **20**, 287–298.

[6] Goldschmidt, R., "Die Sexuellen Zwischenstufen." Monographien aus Dem Gesamtgebiet der Physiologie der Pflanzen und der Tiere, 23 Band, Julius Springer, Berlin, 528 pp., 1931.

[7] Muller, H. J., "Types of Visible Variations Induced by X-rays in Drosophila," *J. Genet.*, **22**, 299–334 (1930).

[8] Dobzhansky, Th., "Genetical and Environmental Factors Influencing the Type of Intersexes in Drosophila Melanogaster," *Amer. Natur.*, **64**, 261–271.

[9] Dobzhansky, Th., "Studies on Intersexes and Supersexes in Drosophila Melanogaster," *Bull. Bur. Gene. Acad. Sci., Leningrad*, **8**, 91–158 (1930).

[10] Patterson, J. T., and T. S. Painter, "A Mottled-Eyed Drosophila," *Science*, **73**, 530–531 (1931).

[11] Dobzhansky, Th., "The Influence of the Quantity and Quality of Chromosomal Material on the Size of the Cells in Drosophila Melanogaster," *Roux' Archiv Entwickl.*, **115**, 363–379.

BODY PATTERN AS RELATED TO MAMMARY GLAND SECRETION

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The effects on body form of the testis as observed in castration—or of the pituitary as noted when it is pathological or experimentally removed as in the experiments of Cushing, Smith, and others, or altered by the animal's heredity as observed in the experiments of MacDowell and Smith—or of the ovary as seen in the work of Goodale on the plumage and structure of birds have made it good form to think of body form as a function of gland secretion rather than gland secretion as a function of body form. The case is reversed, however, when the dairy cattle breeder thinks of his cattle. LeCouteur's paper called attention to this relation of body pattern to milk secretion about 1834. Since that time the relations between body form and milk secretion have grown to include many more points of the growth pattern until more than twenty separate items of the body structure are included in those supposed to affect the amount of milk the cow is able to secrete.[1] Examination of the literature on the subject, however, reveals the fact that in a scientific sense the proof for these supposed relations is largely lacking.

In 1914 the writer commenced a study which has grown to include numerous aspects of this problem. The latest research has about 6000 Jersey cows measured in eight different particulars for its basic data, the measurements being collected by the American Jersey Cattle Club. These measurements include weight, height at withers, depth at withers, heart girth, paunch girth, width at hips, body length, and rump length. Besides these data, the age and parentage of all cows were known from their registration papers. The milk productions and butter-fat yields, etc., were

known on about 1500 with Registry of Merit records. The bearing of these records on the problem of body pattern as related to mammary gland secretion is presented in this paper.

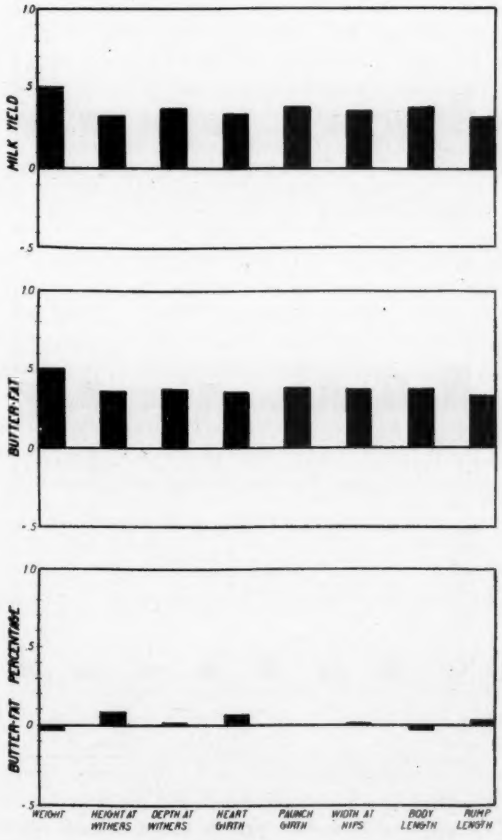

FIGURE 1

Relation between conformation and production of Jersey Registry of Merit cows, all ages.

To fully appreciate this problem it is necessary to approach it in the same manner as the dairyman has done. In the early days little was realized as to the effects of the age of the cow on her milk production and the correlations thus developed between her body form and milk production.

These correlations between age and milk yield or body size are of the order of 0.5. Correlation of such magnitude cannot, of course, be neglected in any true analysis of the problem. However, they were and it remains

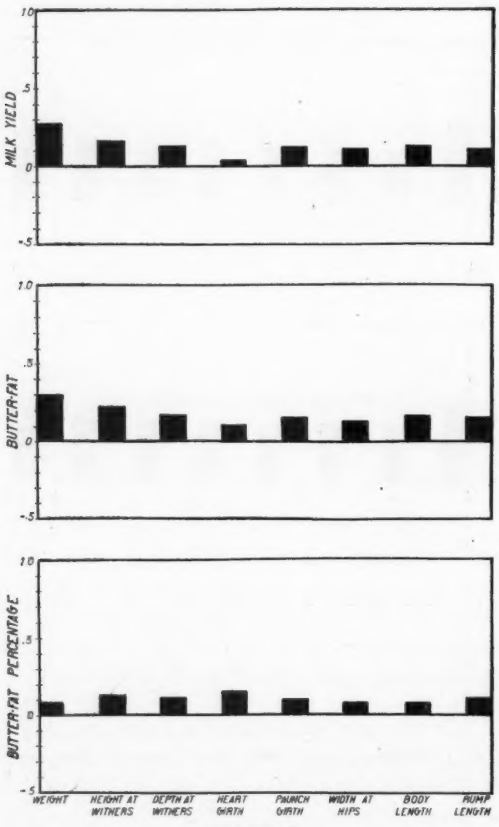

FIGURE 2

Relation between body pattern and mammary gland secretion, influence of age removed by converting all records to their equivalent at the age of 8 years and 3 months.

for us to show their bearing on the observed relations between body pattern and milk secretion. Figure 1 presents the correlation between body size and milk secretion, butter-fat yields, and butter-fat percentage without regard to age. The height of the bars represent value of the correlation coefficients.

The marked influence of body pattern on the quantity of milk secreted and its almost complete lack of effect on the proportion of the constituents within this milk is the most striking feature of data of figure 1. The next most noticeable conclusion is the preponderance of the influence of weight or mass of the cow on the quantity of milk she is able to secrete as contrasted with the other elements of type. The other measurements of the body pattern appear to have equal influence on the quantity of milk which the cow is able to secrete. These correlations are materially influenced by the age of the cow, however. This age effect is removed in the data of figure 2 by correcting each measurement to its corresponding size at the age of 8 years and 3 months.

Comparison of the data of figure 2 with those of figure 1 shows the correlations are reduced to half their previous values by taking into account the influence of age on body size and milk secretion. Age is thus a very important variable in the relation of body form and mammary gland secretion. The next fact to strike the eye is the relation which weight bears to milk secretion as contrasted with that which the bone measurements bear to milk secretion. Weight is considerably more important. This is apparently due to the fact that weight measures not only the skeletal size but also the fleshing which the animal carries on this skeleton.

The next step in the analysis of this problem calls for the determination of the relation between the amount of milk secreted as influenced by the variation of the eight different measurements of body pattern as they separately affect the amount of this secretion. The partial correlation coefficients for milk yield with one of the items, the rest being held constant, furnish a measure of this relation. The correlation coefficients are shown in figure 3.

The data of figure 3, in the light of those in the preceding figures, show that the age of the animal, her weight, and her heart girth are in the order named of significance to milk secretion. The other six measurements of body pattern have insignificant effects. The influence of age on milk secretion has been repeatedly discussed and needs no further amplification. The weight of the cow for her age is of those studied the most important single measurement of body pattern. The larger the cow the larger her milk yield. Weight measures both the degree of fleshing and the general skeletal structure. It follows therefore that both of these items are important to milk secretion. The data are not such as to allow us to discuss the problem of the economics of this production either from the point of view of the protein and calories consumed and returned or the cost in dollars spent and returned. The third significant point is that a definite body pattern favors the secretion of a large quantity of milk. The wedged-shaped appearance of the cow at the shoulders when viewed from the front favors higher milk production than a heavy rounded confor-

mation. This follows from the correlations showing that the cow of relatively small heart girth tends to have the larger milk yield and that

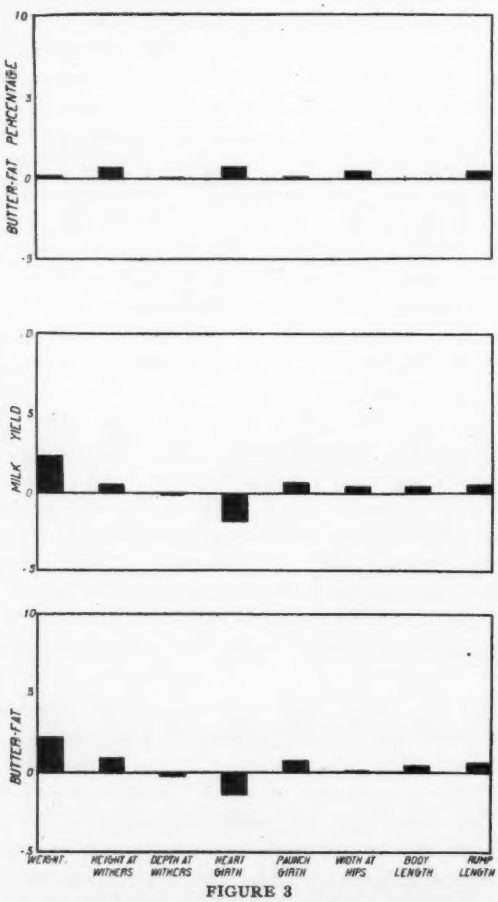

FIGURE 3

Partial correlations between body pattern and mammary gland secretion. Throughout the effect of the given body part on milk secretion as measured is distinct from any influence of the other parts. In all cases age as well as the other seven variables is held constant.

the depth of this girth play has little or no part in milk secretion. The last fact coming from these correlations is the uniform relative unimportance of the other eight points of body pattern.

The negative side of the evidence is of equal interest. It is asserted that the wedge—observable by viewing the cow on the side—bounded by the back line, the hip height, and the lower line of the body is directly indicative of the amount of milk secreted. The fact that the measurements of depth at withers, paunch girth and body length show no marked association with the amount of milk secreted casts grave doubt on this assertion. The fact that the length of rump fails to show any important relation to the amount of milk secreted leads to an adverse conclusion to the oft-repeated assertion that the length of the rump limits the length of the udder and by so doing limits the milk secreted. This feeling of distrust is further borne out by the fact that the width at the hips, another supposed measure of udder size, also shows no important relation to the cow's milk yield. The width at the hips and the body length are functions of the third of the so-called three body wedges which are supposed to be indicative of milk secretion. This wedge is observed in looking down upon the cow. As neither of these measurements has any important association with milk secretion it follows that this wedge has but little significance. Only the quantity of milk secreted or the quantity of the butter-fat are affected by the body pattern. The butter-fat percentage has throughout only insignificant correlations to the body measurements.

The data herein presented allow of the conclusion that two items of body pattern, weight and a wedged-shaped form when viewed from the front are to a limited degree indicative of the amount of milk the cow will secrete. Quantitatively these correlations have a significance comparable to those respectively for the milk yields between half-sisters and single cousins. The other items studied are without significance. None of the items were indicative of the concentration of the butter-fat in the milk.

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WHY CRYSTALS EXIST

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§ 1. *Introduction.*—After the discoveries of Laue and Bragg the physics of crystals progressed very rapidly, both experimentally and theoretically. Taking the existence of regular arrangements of atoms in lattices for granted, Laue, Madelung, Born and others developed the theory of ideal crystals. This theory met with great success as far as certain special properties of crystals are concerned. However, at the same time it became apparent that its foundations were not broad enough to provide for a general scheme in which all the phenomena related to the crystalline state could be incorporated. The purpose of this paper is to show that this failure of the theory may have its origin in the fact that nobody seems to have considered seriously the question why crystals do exist at all. A tentative answer to this question will be given in the following. To obtain this answer it will be found that special emphasis must be laid on the problem of arriving at a clear understanding of the nature of the transitions gas–liquid, and liquid–crystal. The suggested solution also throws new light on the problem of the primary and the secondary (mosaic) structure in crystals.

§ 2. *Difficulties of the Theory of Ideal Crystals.*—One of the greatest puzzles in the physics of solids is the existence of structure sensitive phenomena in single crystals. As an illustration I mention a few of the difficulties which are related to the problem of the mechanical strength of single crystals. (See figure 1.)

(a) The yield point, where a plastic deformation first sets in, lies well within the region of the validity of Hooke's law. It seems fairly obvious that for an ideal lattice a reversible strain-stress curve must be expected, showing deviations from Hooke's law, long before the crystal is actually destroyed.

(b) The yield point in general is much too low inasmuch as a real crystal can be destroyed by forces which are often 100 to 10,000 times smaller than those calculated from the theory of ideal atomic lattices.

(c) The crystals after yielding very often become stronger, sometimes as much as a hundred-fold. This phenomenon, known as cold hardening, constitutes a paradox as it seems impossible to understand why an ideal crystal lattice should become stronger after it has been actually destroyed.

The discussion of difficulties of this kind and the problem of the structure-sensitive properties of solids in general have constituted the main subject of some of my recent publications. I advanced the suggestion that

crystals, geometrically speaking, are not completely characterized by their primary structure alone.[1] On top of the primary structure there appears in general a secondary structure which may be thought of as a crystal within the crystal. The correlation with crystal properties which I have proposed may roughly be characterized by this scheme:

Primary structure————structure insensitive properties

Secondary structure————structure sensitive properties

The idea of the secondary structure provides a satisfactory solution for many of the difficulties encountered by the older theory. Also, a con-

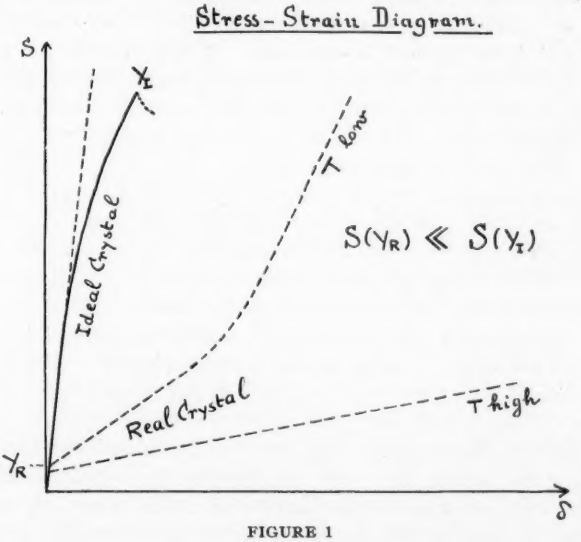

FIGURE 1

siderable number of new phenomena have been discovered, checking predictions made by the new theory. Nevertheless it seems to me that the considerations from which the existence of a secondary structure has been inferred need to be generalized. Indeed, the difficulties mentioned above are of a very general nature and with a few exceptions do not depend on the particular nature of the crystals involved. It appears therefore that a solution of the problem must be found from an understanding of the most general characteristics of a crystal. Now one of the most fundamental properties of a crystal is undoubtedly its existence. The real problem therefore seems to be to understand why crystals exist.

§ 3. *On the Reasons for the Existence of Crystals.*—If we wish to find the physical justification for the existence of crystals we have first to

formulate some of the more outstanding peculiarities of the crystalline state. I think that the following points are important.

(A) There is a sharp melting point which marks the transition from the crystal to the melt. The transition takes place within a very narrow region of temperature. In other words, the crystal and the liquid represent two different phases, thermodynamically speaking. The existence of such a sharp transition point with pressure and temperature constant and only the total internal energy of the system changing, is really a paradox because of the fact that statistical mechanics in general requires a gradual dissociation of atomic complexes with temperature. Indeed, if a system can be in two energy states, E_1 and E_2, the relative probability of the two states is proportional to $e^{(E_2 - E_1)/KT}$, a function which is gradually changing with temperature. It will therefore be necessary to investigate which particular circumstances may cause the dissociation (melting) to take place abruptly. In order to find a solution of this problem we shall have to introduce considerations involving the simultaneous action of great numbers of particles.

(B) Atoms in crystals are lined up over great distances, often to a very high degree of accuracy. This fact is surprising because it is known that the action radii of atoms usually extend over a few Angstroms only. The persistence of direction in a crystal is still more surprising if we remark that it is very little affected by the introduction of relatively large amounts of impurities which certainly change almost completely the fields of force of the atoms making up the original substance. An important consequence of the presence of impurities is that it causes a peculiar coexistence of perfection and imperfection in single crystals. Consider for instance a Zn-crystal. Then, due to the impurities, the slipping strength (yield point) may vary considerably if we progress along the hexagonal axis, whereas the direction of this axis apparently stays unmolested. No similar behavior is known for individual atoms or molecules. Perfection and imperfection seem to appear simultaneously only for larger assemblies of elementary particles.

(C) The transition liquid–crystal is apparently of an entirely different type from the transition gas–liquid. This latter transition can be brought to disappear by a suitable change of pressure. However, nobody has succeeded so far in producing a continuous transition from the crystal to the melt. According to G. Tammann's views such a process is impossible. This is because of the fact that the crystalline state either forms a closed region in the phase diagram or that the melting curve extends to infinity.

Our problem then is to find a conception of the crystalline state which satisfies the requirements A, B and C. We first ask how, from the standpoint of statistical mechanics, it is possible to deduce a sharp transition

point without making use of any trick properties of a special model. It appears that the only relevant phenomena of sufficient generality are those involving the simultaneous coöperation of many particles. If two energy states are involved the Boltzmann factor $e^{-E_1/KT}$ for one state can then be made very large on account of small E_1 and its *a priori* probability very small, whereas for the other state the reverse conditions can be supposed to hold. If we make the total probabilities of the two states equal by choosing a suitable value for the temperature we actually obtain a sharp transition point.

Now there is practically only one possibility to make the (negative) energy of a particular state very large, namely, by choosing coördinates for the different particles which are nearly alike. In other words we must bring the atoms near together. This can, of course, be done both in regard to position and in regard to direction. Suppose that we bring many atoms simultaneously in close contact regardless of any lineup. We then get a very low energy state because of the action of the general cohesive forces. The resulting large Boltzmann factor compensates for the large *a priori* probability which characterizes the substance in the gaseous state. This consideration therefore suggests that the transition gas–liquid is caused by the simultaneous agreement of many atoms to stay close together. It is evident that a transition point of this kind may be made to disappear by a change in pressure which sufficiently cuts down the *a priori* probability of the gaseous state.

The second alternative is to have a great number of articles coöperate not only about their proximity in space but about their lineup as well. If actions of this kind can be found it is very suggestive to make them responsible for the existence of crystals. Such actions would, indeed, satisfy all our three requirements A, B and C. Namely (A), as coöperative actions of many atoms they produce a sharp melting point, (B) they exert an influence over distances which are much larger than the usual atomic dimensions, and therefore are not greatly disturbed by small amounts of impurities and (C), they cannot be essentially influenced by a change of pressure, as this process has no directional characteristics. There is therefore no continuous transition from the crystal to the melt.

On the strength of the above considerations we adopt the following working hypothesis. Sharp transition points and the existence of phases in general are to be attributed to some type of simultaneous coöperation of many elementary particles. If this coöperation refers to the relative mutual distances in space alone, we obtain the transition from the gas to the liquid. If this coöperation refers to the lineup in space also we obtain the transition liquid–crystal.

It is clear that our whole scheme would be worthless unless we can actually find coöperative phenomena in atomic assemblies satisfying the

above requirements. Section 5 will be devoted to a short discussion of some of the more promising possibilities. However, before we go into that, I wish to point out a few general conclusions which are suggested by the previous considerations.

§ 4. *Consequences of the Proposed Hypothesis.*—It appears from the previous section that the usual cohesive forces which act only over atomic distances are essential in effecting the transition from the gaseous to the liquid state. However, a new type of force, extending over many atoms simultaneously, must be investigated in order to arrive at an understanding of the change from the liquid to the crystal. To state it very paradox-

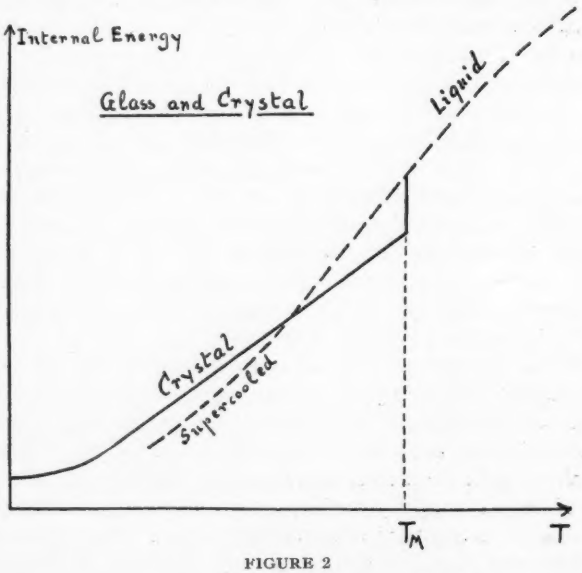

FIGURE 2

ically, we might say, that those forces which determine the primary structure of a crystal are not at the same time responsible for the crystal formation itself. For instance, the primary structure of NaCl is such that each Na^+ ion is surrounded by six Cl^- ions and vice versa. On the other hand in CsCl a Cs^+ ion is surrounded by eight Cl^- ions and vice versa. Our claim is that these arrangements involving six and eight neighbors, respectively, are realized in the melts of the two salts already, at least statistically speaking. These primary arrangements therefore can hardly be made responsible for the crystal formation, inasmuch as they would produce a crystal only at the absolute zero point. If this view is correct it should be possible to find supercooled liquids which

represent configurations of lower energies than crystals of the same substance at the same temperature. The energy diagram of such a substance would be of the type shown in figure 2. We can hope for the realization of this case only if the heat of fusion is very small. Fortunately data for certain glasses are available which prove that liquids may in fact be in a lower energy state than the corresponding crystal.[2] Although only a few cases of this type are known, they are very significant as a support for our point of view.

It follows further that those forces which form the crystal represent an additional amount of energy which must appear in the heat of fusion. Certain deviations of the specific heat curve from Dulong-Petit's law at high temperatures may have their origin here also.[3]

The most interesting conclusion which we can draw from our considerations is that crystals in general must show a secondary structure which is superposed on the primary structure. Indeed, if those simultaneous actions which line up the atoms do not possess the same type of symmetry as the primary structure it will be impossible to obtain a geometrically ideal crystal. How this results in the formation of a secondary structure will be better understood from some examples given in the next section. At any rate we arrive at the conclusion that those elementary phenomena which are responsible for the crystalline state in general also demand the existence of a secondary structure.

We go on to discuss some of the phenomena which involve the simultaneous coöperation of many particles and which therefore represent actions over large distances.

§ 5. *Effects Depending on the Mutual Action of a Great Number of Elementary Particles.*—This is a very brief survey of some phenomena which might be of importance in regard to crystal formation. A more detailed discussion will appear in the *Physical Review*.

(*a*) Permanent Electric Polarization in Crystals.—It is possible to take Herzfeld's theory[4] of the reasons why certain elements are metals and generalize it to cases where not only the polarizability of the atoms is considered but the dielectric properties of atomic or ionic lattices as well. One finds that certain cubic lattices satisfy the requirement of minimum energy only if they are in a state of permanent electric polarization. The criterion whether this will happen or not is furnished by a certain theoretical relation between the exponents by which the distance enters into the expressions for the attractive and the repulsive forces, respectively. The type of secondary structure which may result if the described effect is present will be discussed in another place.

(*b*) Permanent Magnetic Polarization in Crystals.—This phenomenon which characterizes ferromagnetic substances is well known. It causes in iron, cobalt, etc., what might be called a magnetic secondary structure.

This structure is essentially of tetragonal symmetry which in iron is superposed on a cubic primary structure. The elementary particles whose coöperation results in ferromagnetism are electron spins on the one hand and completely saturated blocks containing some hundred thousands of atoms on the other hand.[5] This leads to a rather complicated structure of ferromagnetic crystals inasmuch as one must expect at least two different types of secondary structures which are interwoven with the primary structure. There is some experimental evidence which supports this view.

(*c*) Dehlinger's "Verhakungen."[6]—Suppose that a great number of atoms are arranged on a straight line at mutual distances, *d*, which configuration we assume to be in equilibrium. We then put this assembly into a field whose potential energy has maxima and minima in regular distances, *d*, parallel to our stright line. There are two types of equilibrium configurations. The first is the same as above with the atoms at regular

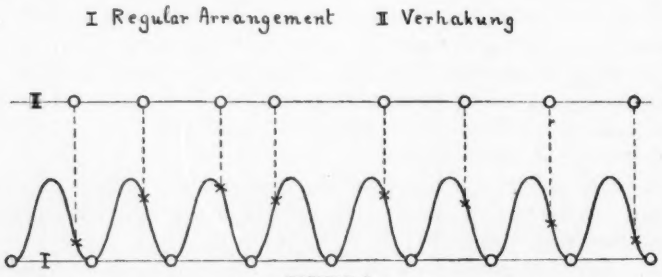

FIGURE 3

distances, *d*. The second equilibrium configuration is obtained by moving two atoms into the same "groove," leaving another groove empty as shown in figure 3. This Dehlinger calls a "Verhakung." It is very important that one such "Verhakung" is dynamically unstable if for the forces involved one assumes values as they actually occur in atomic physics. If, however, one introduces a number of "Verhakungen" successively along the straight line in regular distances D or closer than they stabilize each other. Thermal agitation makes this whole assembly collapse at a definite temperature. In other words there exists a melting point for this assembly.

(*d*) One- and Two-Dimensional Crystals.—There may be other cases than that discussed under (A) where a systematic distribution of electric fields is important. I refer to some of my recent publications (1) which have dealt with the problem of the secondary structure. The essential points were, firstly, the tendency of certain planes or lines of atoms to contract or to expand when isolated, and secondly, the formation of electric fields by assemblies of atoms.

(*e*) Other Types of Secondary Structures.—Finally I wish to mention a few well-known phenomena which suggest the existence of several more types of secondary structures.

In the first place we immediately think of super-conductivity. It seems very tempting to identify the critical point with the melting point of a "crystal within the crystal."

In the second place our attention is drawn to certain transition points which cannot be detected by x-rays. These transitions are nevertheless characterized by markedly abrupt changes of thermal and electrical properties. Such transition points may well turn out to be melting points of certain secondary structures.

In the third place impurities must be considered. If a secondary structure already exists in a pure crystal, the concentrations of foreign atoms must be different for the ideal parts and the regions marking the secondary structure, respectively. But it is also thinkable that the foreign atoms themselves are operative in building up a secondary structure in a given crystal. Foreign atoms can, for instance, exhibit a tendency to form plane crystals or two dimensional lattices which will be stabilized by the original lattice and vice versa. Analogous phenomena on crystal surfaces are well known and have been definitely brought to light by electron diffraction experiments.

§ 6. *Concluding Remarks.*—The purpose of the above considerations was to find a picture of the crystalline state which will allow us to understand the relation between structure sensitive and structure insensitive properties. The following scheme is tentatively proposed.

(1) The ordinary cohesive forces whose action radii are of the order of atomic dimensions are responsible for the condensed states of matter. But they are not characteristic for the crystalline state.

(2) The transition from the liquid to the crystal can only be effected by the simultaneous coöperation of a great number of atoms in regard to a definite directional arrangement.

(3) Several directional phenomena which depend on the coöperation of many particles are briefly described in § 5.

(4) The coöperative phenomenon which is responsible for the stability of a certain crystal will not in general have the same symmetry character as the primary structure. It follows that the crystal as a whole cannot be represented by an ideal crystallographic lattice. A secondary structure must be introduced. The peculiar coexistence of perfection and imperfection which we find in most crystals can easily be derived.

It may be interesting to mention that an analogous persistence of perfection in spite of great external fluctuations is characteristic for living organisms.

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[2] E. Berger, *Forschungen und Fortschritte*, p. 184 (1931).

[3] F. M. Jaeger and E. Rosenbohm, *Proc. Amsterdam Acad. Sci.*, **33**, No. 5, 1930, and **34**, No. 1, 1931.

[4] K. F. Herzfeld, *Phys. Rev.*, **29**, 701 (1927).

[5] F. Bitter, *Ibid.*, 91 (1931).

[6] U. Dehlinger, *Ann. Physik*, **2**, 786, 1929.

ON THE AIR RESISTANCE OF PROJECTILES

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1. *Introductory.*—In making this study, the author had in mind its applications to the problem of meteoric impact. The different laws of resistance, heretofore proposed, contain too many empirical or hypothetical elements to permit a safe extrapolation to cosmic velocities. A more thorough investigation into the theoretical aspects of the problem seemed, therefore, desirable. It is known from the experimental results of aeronautics and hydraulics and from their theoretical analysis, that, for moderate velocities, already, the influence of viscosity is small compared with that of inertia. It seems, therefore, permissible to neglect the viscous terms altogether even in the range of velocities attained by ordinary projectiles. The problem is reduced, by this remark, to finding solutions of Euler's equations of hydrodynamics for velocities higher than the velocity of sound. While looking about for a simple case in which the solution could be carried through, it occurred to the author that in the two-dimensional case certain well-known integrals of Euler's equations due to Riemann,[1] Prandtl[2] and Th. Meyer[3] permit a rigorous solution of the problem for polygonal contours. For simplicity, we take a quadrangle symmetrical with respect to one of its diagonals and moving in the direction of this diagonal (Fig. 1). We shall find the rigorous expression for the resistance of such a body in section 7. For lower velocities the formulae are rather involved, but for the limit of very high velocities they are reduced to the very simple expression (28) which remains valid for contours with curved sides and for three dimensional bodies. The properties of our solution and its relation to some laws of ballistic resistance proposed by others and to experiment are discussed in section 8.

2. *Some Mathematical Aspects of the Problem.*—The case of high velocity is in some respects simpler than that of low velocity. The front point

A of the projectile is moving with a velocity higher than the velocity of sound. Therefore, the air to the left side of it has no intimation of the projectile's approach and remains at rest up to the very point A. The line separating the region of resting air from that of moving air is obtained by a simple reasoning due to Mach.[4] At every moment ($t = 0$) the point A can be regarded as the source of a spherical wave. After the lapse of time t, this spherical wave will have expanded to the radius $R = at$, where a is the velocity of the wave front under the particular conditions of the problem. In the same time the point A will have progressed to a new position A' through the distance $l = wt$. It is easy to see that a straight line, going out from the point A' and tangent to the circle of radius R with its center in A, is the envelope of all circular waves starting from the front point in its different positions.[5] This line is, therefore, the line of discontinuity separating the two regions of resting and of moving air.

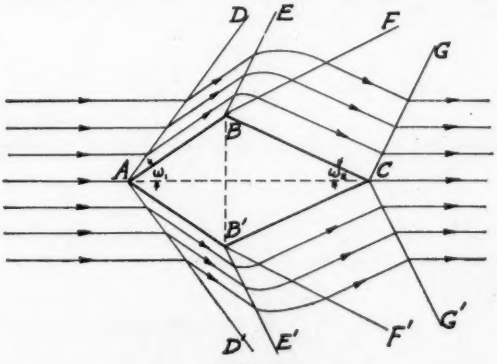

FIGURE 1

Its angle of inclination α to the direction of motion is given by the simple relation $\sin \alpha = R/l$ or

$$\sin \alpha = a/w, \tag{1}$$

and is called Mach's angle.

Quite similar conditions obtain in the case of a current streaming along a plane wall. If the velocity of this current is higher than that of sound, any sudden break in the wall, or other irregularity, cannot produce any effect on the character of motion in a region limited by a straight line going out from this irregularity and including Mach's angle with the current. These conditions have the consequence that the air filled space around our polygon (Fig. 1) breaks up into a number of sections separated by straight lines of discontinuity. The complete solution can then be pieced together from two simple solutions given by Riemann and by Prandtl. We give these two solutions in the sections 3, 4 and 5.

3. *Riemann's Shock Wave.*—It was shown by Riemann that in a compressible medium there can exist surfaces of discontinuity moving with a certain velocity in the direction of their own normal. Let us regard a plane discontinuity or "*shock wave*" from the point of view of the observer moving with it, so that the shock wave will appear to us stationary. We denote pressure, density and normal velocity of the medium, to one side of the discontinuity, by p_1, ρ_1, v_1, to the other side of it, by p_2, ρ_2, v_2. It will be convenient to regard the discontinuity as the limiting case of a very thin layer in which the corresponding quantities are denoted by p, ρ, v and in which they change continuously from p_1, ρ_1, v_1 to p_2, ρ_2, v_2. Let us further denote the total velocity by w and the component parallel to the inhomogeneous layer by u. We start from Euler's equations for a two dimensional stationary electrodynamical field:

$$\left.\begin{aligned} \rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) &= -\frac{\partial p}{\partial x}, \quad \rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}, \\ \nabla\cdot(\rho\bar{w}) &= \frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y} = 0. \end{aligned}\right\} \quad (2)$$

We are interested only in irrotational solutions of these equations

$$\bar{w} = -\nabla\psi. \quad (3)$$

Owing to this form of $\bar{w}$, there exists the relation $\partial u/\partial y = \partial v/\partial x$, and the first two equations can be combined into the well-known relation of Bernoulli

$$\tfrac{1}{2}dw^2 + \frac{dp}{\rho} = 0. \quad (4)$$

Let y be the direction normal to the inhomogeneous layer. Then ρ is independent of x while u is constant: the third eq. is, therefore, reduced to $\partial(\rho v)/\partial y = 0$ or

$$\rho v = \text{const.} \doteq \rho_1 v_1 = \rho_2 v_2, \quad (5)$$

while eq. (4) becomes $\frac{1}{2}dv^2 + dp/\rho = 0$. Substituting v from eq. (5), we obtain

$$dp = \rho_1^2 v_1^2 d\rho/\rho^2. \quad (6)$$

The integral of this relation taken across the layer gives

$$p_2 - p_1 = v_1^2(\rho_2 - \rho_1)\rho_1/\rho_2. \quad (7)$$

This is the fundamental equation of Riemann's shock wave giving the difference of pressure on both sides of the discontinuity in terms of the densities.

We are going to apply these relations to the motion in a concave angle formed by two plane walls (Fig. 2). The shock wave separates two regions of the medium characterized by ρ_1, p_1 and ρ_2, p_2, respectively. The total velocities w_1 and w_2 are parallel to the walls BA and AC. The components normal to the shock wave v_1, v_2 are subject to conditions (5), (7), while the tangential component u is continuous and has the same value on both sides. We have the relations

$$\left.\begin{aligned} u &= w_1 \cos\alpha, & v_1 &= w_1 \sin\alpha, \\ u &= w_2 \cos(\alpha - \omega), & v_2 &= w_2 \sin(\alpha - \omega). \end{aligned}\right\} \tag{8}$$

We can eliminate w_1 and w_2 from these formulae. If we take into account equation (5) and denote, for short, $\rho_2/\rho_1 = x$, the result of this elimination is $tg\alpha = xtg(\alpha - \omega)$ or

$$cotg\ \alpha = \left[x - 1 \pm \sqrt{(x-1)^2 - 4xtg^2\omega}\right]/2xtg\omega. \tag{9}$$

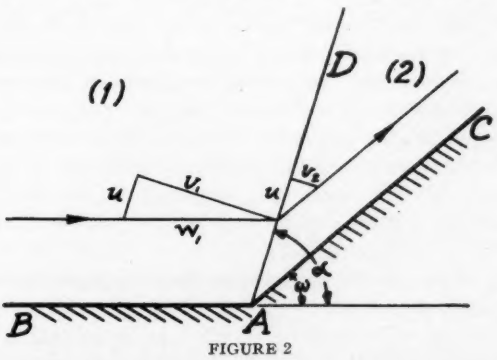

FIGURE 2

This equation gives us the angle of inclination α of the shock wave as a function of the condensation x. It is interesting to note that there are two possible solutions of the problem, corresponding to the two signs of the square root. There is no mathematical reason to prefer one solution to the other, but there is a dynamical one which rules out the minus sign. We shall, therefore, omit this sign in the following formulae and we shall return to the physical discussion of our reasons for this in section 8. On the other hand equation (7) can be thrown into the form

$$\frac{p_2 - p_1}{\rho_2 - \rho_1} = \frac{\rho_1}{\rho_2} w_1^2 \sin^2\alpha. \tag{10}$$

Substituting α from the preceding equation, we obtain

$$\frac{\rho_1 w_1^2}{p_1} = \left[\frac{x - 1 + \sqrt{(x-1)^2 - 4xtg^2\omega}}{2xtg^2\omega} + 1\right]\cdot\left(\frac{p_2}{p_1} - 1\right), \tag{10'}$$

a formula giving the relation between the original velocity w_1 and the condensation x, if the law connecting pressure and density

$$p = f(\rho) \tag{11}$$

is known.

4. *Pressure-Density Relation in a Shock Wave.*—There seems to exist little clarity as to the form which equation (11) has in a shock wave. Most authors have used the adiabatic relation $p = k\rho^\gamma$, however, it is obvious that the conditions of compression in a shock wave are vastly different from those in a stationary vessel, so that there is not much justification for this law. Let us set up the equation of the energy balance for the shock wave. We follow a unit of mass of air in its motion across the discontinuity and assume that the total energy change in it is equal to nothing.

$$U_2 - U_1 + \tfrac{1}{2}(w_2^2 - w_1^2) + W = 0. \tag{12}$$

We denote by U the internal energy of the gas, by W the work which it does in moving from one side of the discontinuity to the other. This work consists of two items: on the one hand, it must push away a volume of gas equal to its own ($V_2 = 1/\rho_2$) on the front side, on the other hand, it vacates the volume $V_1 = 1/\rho_1$, on the back side, allowing the gas behind it to do work against it. The balance is, obviously,

$$W = p_2V_2 - p_1V_1 = \int_1^2 pdV + \int_1^2 Vdp. \tag{12'}$$

The increase of kinetic energy is known from equation (4) to be $\frac{1}{2}\,dw^2 + Vdp = 0$, which gives

$$\tfrac{1}{2}(w_2^2 - w_1^2) + \int_1^2 Vdp = 0.$$

Subtracting it from (12), we find

$$U_2 - U_1 + \int_1^2 pdV = 0. \tag{13}$$

This equation is satisfied by the adiabatic law of compression, and this may have been the reason why this law was used by many writers on the subject. However, there is an infinity of other laws satisfying the same relation and the adiabatic hypothesis leads to grave contradictions with the rest of our formulas. To find the true law we have only to replace the discontinuity by an inhomogeneous layer as in section 3. The dependence of p on ρ is given there in equation (6) which we shall write in its integrated form

$$p = p_1 + \rho_1^2 v_1^2 \left(\frac{1}{\rho_1} - \frac{1}{\rho}\right). \tag{14}$$

Supposing that the equation of state of our gas is sufficiently well represented by $pV = RT$, and the internal energy by $U = C_v T = pV/(\gamma - 1)$, we find from (12), (12′) and (5)

$$2\gamma p_1 + \left[\gamma - 1 + \frac{\gamma + 1}{x}\right]\rho_1 v_1^2 = 0.$$

Eliminating v_1 between this equation and (7), we arrive at the final law

$$p_2 = \frac{(\gamma + 1)x - (\gamma - 1)}{(\gamma + 1) - (\gamma - 1)x}\, p_1. \tag{15}$$

This leads to the following expression for the change of internal energy

$$U_2 - U_1 = C_v(T_2 - T_1) = \frac{x^2 - 1}{[\gamma + 1 - (\gamma - 1)x]x}\,\frac{p_1}{\rho_1}. \tag{16}$$

It is easy to check that expressions (14) and (16) really satisfy the general condition (13). The law (15) is quite different from the adiabatic law $p_2 = x^\gamma p_1$, and this means that inside the inhomogeneous layer, while the compression is going on, there is conduction of heat from element to element of the gas. However, every element loses as much energy in the second half of its path through this layer as it gains in the first half because, without this, a stationary flow could not be maintained. *It is assumed that there is no appreciable loss of energy, by conduction and radiation, to bodies outside the wave.* This assumption is sound enough as long as the compressions and temperature differences are not too high. We are going to discuss its limitations and the influence of possible deviations from it in section 9. For the present, however, we shall base ourselves on this assumption and on formula (15) which is its mathematical expression. Substituting into (10′), we obtain our final formula

$$\frac{w_1^2}{a_1^2} = \left[\frac{x - 1 + \sqrt{(x-1)^2 - 4x\mathrm{tg}^2\omega}}{2x\mathrm{tg}^2\omega} + 1\right]\cdot\frac{2(x-1)}{(\gamma+1) - (\gamma-1)x}, \tag{17}$$

where $a^2 = \gamma p/\rho$ is the square of the velocity of sound. This relation establishes the connection between the original velocity w_1 and the condensation x. As an immediate consequence of it, we can obtain the pressure p_2 by means of equation (15). If we wish to find w_2, we have to compute α from equation (9) and to use the relation $w_2 = w_1 \cos\alpha/\cos(\alpha - \omega)$ which follows from equation (4).

5. *The Prandtl-Meyer Solution.*—This solution was published in *Th. Meyer's* thesis which is, at present, almost inaccessible. It is, therefore, well to give here a detailed exposition of it. Meyer's own derivation is somewhat cumbersome, and we present it here in a simplified form.

We start from Euler's equations in two dimensions in the form (2), (3), (4) of section 3. In polar coördinates the equations of energy and of continuity acquire the form

$$\frac{p}{2}\,d\left[\left(\frac{\partial\psi}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\psi}{\partial\varphi}\right)^2\right]+dp=0,$$
$$\frac{\partial}{\partial r}\left(r\rho\,\frac{\partial\psi}{\partial r}\right)+\frac{1}{r}\,\frac{\partial}{\partial\varphi}\left(\rho\,\frac{\partial\psi}{\partial\varphi}\right)=0.$$

We restrict our problem further by looking only for states of motion, in which the velocity $\bar{w}$ and the density ρ are independent of r and are functions of the angle φ only. This will be the case when the potential has the form

$$\psi = r\chi(\varphi)$$

leading to the formulae

$$\left.\begin{aligned}\frac{\rho}{2}\,d\left[\chi^2+\left(\frac{d\chi}{d\varphi}\right)^2\right]+dp&=0,\\ \rho\chi+\frac{d}{d\varphi}\left(\rho\,\frac{d\chi}{d\varphi}\right)&=0.\end{aligned}\right\} \tag{18}$$

Multiplying the last equation by $d\chi$ we obtain

$$\frac{\rho}{2}\,d\left[\chi^2+\left(\frac{d\chi}{d\varphi}\right)^2\right]+\left(\frac{d\chi}{d\varphi}\right)^2 d\rho=0. \tag{19}$$

Comparing this with equation (18) we find

$$v_\varphi{}^2=\left(\frac{d\chi}{d\varphi}\right)^2=\frac{dp}{d\rho}=f'(\rho)=a^2. \tag{20}$$

The azimutal component v_φ of the velocity is always equal to the velocity of sound. This shows, among other things, that Prandtl's solution applies only to velocities higher than that of sound. Substituting (20) into (19), we obtain a relation between χ and ρ

$$\chi^2+f'(\rho)+2\int\frac{f'(\rho)}{\rho}\,d\rho=C.$$

Differentiating this with respect to φ and remembering that $d\chi/d\varphi=\sqrt{f'(\rho)}$, we arrive at a plain differential equation for ρ. There are no objections against the adiabatic law $f(\rho)=k\rho^\gamma$ in this case: in the absence of conduction and radiation, it is the correct connection between pressure

and density. Assuming this law, the equation can be readily integrated, giving

$$\left.\begin{aligned} \psi &= r\sqrt{C}\sin\sqrt{\frac{\gamma-1}{\gamma+1}}\,(\varphi-\varphi_0), \\ \rho &= \left[\sqrt{\frac{C(\gamma-1)}{k\gamma(\gamma+1)}}\cos\sqrt{\frac{\gamma-1}{\gamma+1}}\,(\varphi-\varphi_0)\right]^{\frac{2}{\gamma-1}}. \end{aligned}\right\} \tag{21}$$

6. *Applications.*—We are now in a position to solve the problem of the flow around a corner (Fig. 3). Let the initial motion have the velocity w_2 parallel to the plane AO. According to our discussion in section 2, this type of motion will extend until the plane OB going out from O under Mach's angle:

$$\sin\beta_1 = a_2/w_2,$$

so that the angle β_1 can be considered as known. Beyond OB the type

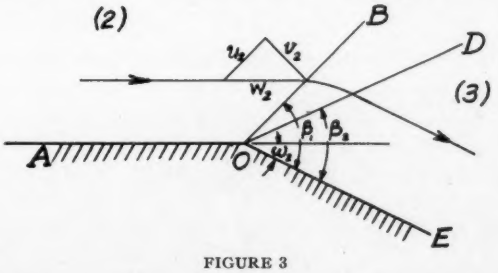

FIGURE 3

of motion will be that of Prandtl's solution (21). The two parameter's C and φ_0 must be chosen so as to establish the continuity of the velocity across the line OB. If we call the radial and azimutal components in Prandtl's case u_r and v_φ, we must have, for $\varphi = \pi - \beta_1$, the conditions $u_r = w_2\cos\beta_1$, $v_\varphi = w_2\sin\beta_1$. With the help of (3) and (21) this leads to a determination of φ_0 and C by the following conditions

$$\left.\begin{aligned} \operatorname{tg}\beta_1 &= \sqrt{\frac{\gamma-1}{\gamma+1}}\operatorname{cotg}\sqrt{\frac{\gamma-1}{\gamma+1}}\,(\pi-\beta_1-\varphi_0), \\ C(\gamma-1) &= w_2^2(\gamma-\cos 2\beta_1). \end{aligned}\right\} \tag{22}$$

From the line OB on, the path of the particles will be curved. The direction of velocity will bend round until it becomes parallel to the second plane OE of the corner. This will occur in a second straight line OD whose angle of inclination is given by a relation analogous to (22)

$$\operatorname{tg}(\beta_2+\omega_2) = \sqrt{\frac{\gamma-1}{\gamma+1}}\operatorname{cotg}\sqrt{\frac{\gamma-1}{\gamma+1}}\,(\pi-\beta_2-\varphi_0). \tag{23}$$

Because of relation (20), the line OD will be inclined under Mach's angle to the plane OE. Beyond OD we have another region of parallel flow with velocity w_3 determined by $u_r = w_3 \cos \beta_2$, $v_\varphi = w_3 \sin \beta_2$, for $\varphi = \pi - \beta_2$. The pressure in this region is constant and the same as in the line $\varphi = \pi - \beta_2$, which can be obtained from (11) and (21).

However, the type of flow around a corner, represented by Fig. 3 and described in the preceding paragraphs is not quite general. This becomes clear when we write out the equation of a stream line in Prandtl's solution, which is easily obtained from expression (21) for the potential. This equation becomes

$$r^{\frac{\gamma-1}{\gamma+1}} \cos \sqrt{\frac{\gamma-1}{\gamma+1}} (\varphi - \varphi_0) = C',$$

it represents a loop with its nearest point to the origin in $\varphi = \varphi_0$, going into infinity in the directions $\varphi - \varphi_0 = \pm \sqrt{\frac{\gamma+1}{\gamma-1}} \frac{\pi}{2}$. (If we take $\gamma = 1.4$, this angle is $\pm 220°27'$.) We see, therefore, that the angle by which the velocity can be turned is limited. Under certain conditions we may have a type of motion different from that of Fig. 3. As in that case, we shall have parallel flow up to the line OB and Prandtl's solution beyond it. However, the bend of the velocities may reach its limit before they become parallel to the plane OE. In this case, the stream will go off under an angle to OE leaving a wedge like sector of dead space between itself and this plane. This wedge will have pressure and density equal to zero: it will be pumped free of air by the suction of the current. It is not hard to see that the condition for this type of motion is

$$\omega_2 > \sqrt{\frac{\gamma+1}{\gamma-1}} \operatorname{arctg} \left(\sqrt{\frac{\gamma+1}{\gamma-1}} \operatorname{tg} \beta_1 \right) - \beta_1. \qquad (24)$$

7. *Flow Around a Quadrangle.*—The formulae developed in sections 3, 5, 6 suffice to give an answer to our problem: the resistance of a quadrangular contour. The original current (Fig. 1) with the velocity w_1 is limited by the shock wave and is converted into a flow of the velocity w_2 parallel to the side AB. This velocity and the pressure p_2 on the plane AB can be computed by means of the formulae of section 4. These data determine also the position of the line BE which represents the limit of this type of motion. In the wedgelike space between BE and BF applies the Prandtl-Meyer solution. Our expressions of section 6 permit us to determine the position of the line BF and the velocity and pressure obtaining in this line. This pressure p_3 is the same as that which prevails in the whole region of parallel flow beyond BF and which is applied

to the back plane BC. Finally, the flow encounters a second shock wave CG, but since we already possess all the data relating to the question of resistance, we need not discuss this last process in detail.

It is obvious that the resistance per unit length of our prism of section $ABCB'A$ is given by the expression

$$F = (p_2 - p_1)S, \tag{25}$$

where S is the length of the diagonal BB' which we shall call the *cross section of the projectile.*

We compute the resistance for some typical cases. Let us take as initial velocity $w_1 = 3.36a_1$, i.e., 3.36 times the velocity of sound in a medium having $\gamma = 1.40$ and the pressure p_1. (For instance, in air under normal conditions: $a_1 = 330$ m/sec, $w_1 = 1110$ m/sec). The results can be summarized in the following table.

$2\omega_1$	$2\omega_2$	p_2	p_3	F/S	$\gamma_1 w_1^2 \sin^2 \omega_1$	ρ_2/ρ_1
60°	16°40′	8.00	0.86	7.14	3.95	3.50
60°	60°	8.00	0.14	7.86		
16°40′	16°40′	1.95	0.74	1.21	0.33	1.60
16°40′	60°	1.95	0.07	1.88		

Pressures and F/S are expressed in units p_1 (atmospheres). We see that the resistance has its highest value when the projectile is blunt from both ends, it is lowest when it is sharp from both ends. If the rear end is very abrupt, p_3 vanishes altogether.

As a second example, let us take a projectile with a front angle of $2\omega_1 = 29°$ and an abrupt rear end. As we shall see in section 10, we can interpose between the front part and the rear part any length of a parallepiped, parallel to the direction of motion, without changing the resistance. The computations are quite easy in this case, because the pressure on the back planes vanishes and we need only apply one formula, viz., equation (17) of section 4. This formula gives us the compression x for every velocity w_1, while the pressure $F/S = p_2$ follows from formula (15).

w_1 in m/sec	580	600	700	800	900	1000	1100	1200	1500
F/S	2.08	2.10	2.18	2.35	2.54	2.76	2.99	3.24	4.10
F/Sw_1^2	6.20	5.84	4.45	3.68	3.13	2.76	2.43	2.24	1.83

w_1 in km/sec	2.0	2.5	3.1	4.0	5.26	9.0	13.1	34.7
F/S	5.79	8.10	11.5	17.3	29.0	81.5	169	1160
F/Sw_1^2	1.44	1.29	1.19	1.08	1.05	1.01	0.99	0.96

The theoretical limit for the ratio F/Sw_1^2 is 0.94. It should be pointed out that the current flows along the back surface only when w_1 is quite low. For higher velocities it detaches itself and forms a wedge of dead space at the rear side of the projectile. According to equation (24) the

velocity at which the wedge extends into infinity is 6790 m/sec. For higher values of w_1 the currents, coming from the upper and the lower side, diverge leaving between them an open sector of dead space. As this dead space has the pressure zero, the result seems paradoxical: the question arises how it is possible to evacuate an infinite portion of space. This difficulty will be resolved in section 10.

8. *Discussion of Results.*—An outstanding characteristic of our formulae is that they apply only to velocities w_1 above a certain lower limit. This appears, clearly, from expressions (9) and (17): for every given angle there exists a certain minimum value of x below which the condensation cannot fall without making the square root of these expressions imaginary. This value turns out to be $x_{\min.} = (1 + \sin \omega)^2/\cos^2 \omega$, the corresponding value of α is $\alpha = 45° + \omega/2$. The minimum velocity is then obtained from equation (17). For instance, we have[6]

ω	$x_{\min.}$	$\alpha_{\max.}$	$w_{\min.}$
8°20′	1.339	49°10′	520 m/sec
14°30′	1.668	52°15′	580 m/sec
30°	3.000	60°	760 m/sec
45°	5.829	67°30′	1090 m/sec

As the front angle increases the minimum velocity becomes higher and higher. This means that the type of motion, represented in Fig. 1 and characterized by a shock wave consisting of two separate branches AD and AD' can develop only when the velocity w_1 is above this limit. For velocities below $w_{\min.}$ and above the velocity of sound we should expect a front discontinuity of a different type (probably a single branched curved line not passing through the point A). If we plot the resistance against the velocity, we must distinguish three different parts of the curve: For velocities under that of sound ($w_1 < a_1$) we have a type of flow around the projectile without any discontinuities and, presumably, not very different from the potential flow. For velocities between a_1 and $w_{\min.}$ a type of motion with a discontinuity of an unknown form. For $w_1 > w_{\min.}$ the conditions which we have analyzed in the preceding sections. These conclusions, though derived from considering a two dimensional model are completely borne out by observations with actual three dimensional projectiles. E.g., in *Cranz'* book[7] we find on pp. 64 and 65 curves referring to bullets with sharp points. The quantity F/Sw_1^2, is here plotted against the velocity w_1 and the three regions are very conspicuous. For $w_1 < a_1$, the curve is almost horizontal, then we have a narrow region of a steep rise to a maximum which lies between 500 and 600 m/sec, corresponding to our $w_{\min.}$, and beyond this a monotonic and slow decline. Our result, that $w_{\min.}$ increases with the front angle, is also confirmed: on p. 62 some data for cylindrical projectiles (with flat

front) are given and the maximum of the curve is not reached within the velocities of experimentation (1300 m/sec).

Of the ballistical formulas for resistance proposed heretofore, those given by Sommerfeld[8] and by H. Lorenz[9] are partially based on theoretical considerations. The fundamental difference between these laws and ours is that they attempt to describe the resistance for any velocity w_1 by a single formula. On the contrary, we maintain that in the three regions of velocity $w_1 < a_1$, $a_1 < w_1 < w_{\text{min.}}$, $w_1 > w_{\text{min.}}$, we must have different analytical expressions, since we have in them different types of hydrodynamical flow. Our solution applies only to the last region, and here it represents the observations in a satisfactory way. On the one hand, our formula which *does not contain any empirical constants gives the right order of magnitude* for the resistance. Our table at the end of section 7 shows that, for velocities of about 1000 m/sec, the resistance is about 2.8 atm/cm^2. Cranz and Becker (loc. cit., p. 63) measured for rifle bullets at the same velocity 1.8 atm/cm^2. On the other hand, the qualitative trend of our theoretical curve is the same as that of the curve given by Cranz. The quantity F/Sw_1^2 decreases and approaches asymptotically a constant value. That the decay in our table (section 7) is somewhat slower than in Cranz' curves, is without significance: the resistance depends in our theory on the front angle $2\omega_1$, and our choice of 29°, as an illustration, is purely accidental. The only discrepancy is the fact that our theoretical curve for F/S starts with a horizontal tangent for $w_1 = w_{\text{min.}}$ and then bends upward, while the experimental curve has an upward trend also in the beginning of the region. This may be a characteristic property of our two-dimensional model which, of course, cannot give a quantitative agreement with observations upon three dimensional projectiles.

It is here the place to return to the question of the existence of a second solution, touched upon in section 3, and to explain why we did not attribute to this solution a physical reality. As we see from equation (9) the two solutions differ by the sign of the square root. For $x_{\text{min.}}$ (and $w_{\text{min.}}$) the square root vanishes and the two solutions coincide. We have pointed out that the angle α between the shock wave and the velocity w_1 has then the value $\alpha = 45° + \omega/2$. As the compression x increases, α becomes smaller in the first solution and larger in the second, the extreme values, for very large x, being $\alpha = \omega$ and $\alpha = \pi/2$, respectively. The velocity w_2 in region (2) of figure 2 is given by $w_2 = w_1 \cos\alpha/\cos(\alpha - \omega)$. For a given w_1, it is smaller in the second solution than in the first. In fact, in the second solution, w_2 drops to values under the velocity of sound, for quite moderate velocities w_1, and continuous decreasing as w_1 goes up. This solution is, therefore, characterized by a higher potential energy of compression and by a lower kinetic energy than the first solution

for the same w_1. Consequently, also the Lagrangean function and the principal function will be higher in it. According to Hamilton's principle every dynamical state tends to a minimum of the principal function, so that we conclude that the second solution is unstable and we have to deal only with the first.[10]

9. *The Results Do Not Materially Depend on Special Assumptions.*— The formulas of the preceding sections are based on the assumption that the compression and expansion of air in its wave motion takes place without loss of energy by conduction and radiation. We may expect that this assumption breaks down when the temperature in parts of our wave becomes high, i.e., when the density ρ_2 and the velocity w_1 are large. Under these conditions the resistance depends almost exclusively upon the shock wave. The main question that must be answered is, therefore: how does the pressure p_2, in the shock wave, change, if we have deviations from the law (15) connecting pressure and density? It can be readily seen from equations (9) and (10) that these changes are very small, indeed. In the limiting case of extremely high compressions ($\rho_2 >> \rho_1$, $x >> 1$), these equations give $\alpha = \omega$

$$p_2 - p_1 = \rho_1 w_1^2 \sin^2 \omega. \tag{26}$$

If the compression is fairly large and ω not over 45°, this expression gives an approximation. It is entirely independent of the law of compression, so that this law affects only small correction terms. The writer took the trouble of computing the resistance, for the whole range of velocities w_1, using the adiabatic law instead of formula (15). Though the compression x corresponding to every velocity w_1 is widely different in the two cases, the relation between F/S and w_1 is, practically, the same. This is the reason why the adiabatic law could be used with comparative impunity.

The temperature T_2 in the shock wave can be obtained from equation (16) and goes up to rather high values:

w_1 in km/sec	1	2	3	5	9	19	35
T_2 (Cent.)	104°	245°	500°	1300°	3700°	15,600°	53,000°

The energy losses by conduction and radiation must be, therefore, considerable. In fact, T_2 is asymptotically proportional to w_1^2, while the time of contact with the projectile is inversely proportional to the first power of w_1. The relative importance of conduction increases as w_1, that of radiation at a much faster rate. Although this fact does not materially affect the law of resistance, it is interesting to discuss its implications. Formula (17) seems to indicate that there is an absolute maximum for x and for ω. In fact, when $x = (\gamma + 1)/(\gamma - 1)$, $w_1 = \infty$; on the other hand, when $\mathrm{tg}\omega > 1/\sqrt{\gamma^2 - 1}$, the square root is imaginary

for all permissible values of x. But owing to conduction and radiation these limitations have no physical reality. The real temperature will be much lower than that computed from formula (16) and, therefore, the compression x much higher (since p_2 remains, practically, unchanged). Because of this, our theory will be applicable to any front angle up to the neighborhood of 90°, and formula (26) will represent a very close approximation of the real pressure at high velocities.

The thermic energy losses will affect also the character of the Prandtl-Meyer expansion waves. Here, too, we must expect, for very high velocity w_1, much smaller differences of temperature and pressure than would follow from the adiabatic law. In the case of isothermic expansion, we should have, instead of equations (14)

$$\psi = \sqrt{RT}\, r(\varphi - \varphi_0),$$

$$\log \rho = \frac{C}{RT} - 1 - \frac{(\varphi - \varphi_0)^2}{2}.$$

The equation of a stream line would take the form

$$r = C' \exp \tfrac{1}{2}(\varphi - \varphi_0)^2.$$

This equation shows that there is no limiting angle for the deflection as in the case of the adiabatic law. The stream lines around a convex angle, under all circumstances, will be parallel to the sides and a dead space will never be formed.

With increasing velocity w_1, the conduction and radiation begin to dominate the heat exchange and the actual flow begins to resemble these conditions closer and closer. The dead space becomes narrower and narrower and in the limit ($w_1 = \infty$) the flow forms only a thin film on the surface of the polygon.

10. *Generalizations and Limiting Values.*—In the preceding sections we have computed the resistance for quadrangles. The same method leads itself to the treatment of convex polygonal contours which may be chosen symmetrical or asymmetrical with respect to the direction of motion. We have, in this case, shock waves, in the front and in the rear, and Prandtl wedges at all the remaining corners. It is obvious, from the character of Prandtl's solution, that the pressure on any side of the polygon depends only on the two angles which this side and the front side include with the direction of motion. It is independent of the position and length of all the intermediate sides of the polygon. This circumstance permits the use of our formulas of sections 4 and 5 for computing the pressures on all sides of the polygon: we have only to take as ω_2 the angle between the front side and the side we are considering. We can say, therefore, that the solution for a polygon with any number of sides is explicitly contained in our formulas.

However, in extending the method to polygonal contours, one should restrict himself to cases when none of the angles ω_2 do satisfy the inequality (24). This precaution is necessary for the following reason: The lines AB and BE of figure 1 intersect, if continued. The region of flow parallel to AB is limited to a finite triangle. Though we have no accurate expression for the state of motion above this triangle, it is safe to say that the influence of the projectile will not extend far beyond the point of intersection. At a certain distance beyond this point we shall find again the original parallel flow, as it exists to the left of the point A. If our contour is such that we have, in addition to AB, other sides sticking out above the point B, it may happen that one of these sides will be also above the point of intersection of AB and DE and will receive its pressure from a portion of the fluid which has not passed through the shock wave AB. While restricting the application of the method, this consideration removes the difficulty mentioned at the end of section 6. The state of motion analyzed in the preceding sections is restricted to a strip of finite width behind the projectile and, consequently, the energy necessary to maintain the motion is also finite.

Of particular interest to the writer is the limiting case of very high velocities w_1. We have seen in the preceding section, that in this case the compression x is very large and that the angle α approaches the limiting value $\alpha = \omega_1$. Neglecting in formula (26) p_1 beside p_2 we obtain the law of resistance

$$F/S = \rho_1 w_1^2 \sin^2 \omega_1. \tag{27}$$

This law can be considerably generalized. What will happen, if we have a polygon instead of the quadrangle of Fig. 1? With increasing velocity w_1 the intersection of AD and BE comes nearer and nearer to the point B, and in the limit $w_1 = \infty$, it strictly coincides with it. The whole triangle is reduced to an infinitesimal film covering the side AB. We have seen, also, that at the same time the current turns completely in the wedge adjacent to BE and becomes parallel to the next side. It follows that this side, which we suppose to be sticking out beyond AB is completely exposed to the original parallel flow, exactly like the side AB itself. Consequently, the pressure on this side can be computed by the same formula (27). The resistance of a polygon is, therefore, given by the formula

$$F/S = \rho_1 w_1^2 \overline{\sin^2 \omega}. \tag{28}$$

where $\overline{\sin^2 \omega}$ is the mean squared sine of the angles which the different sides of the polygon include with the velocity w_1. The average must be taken only over sides turned toward the current (front half of the polygon),

as the pressure on the back half vanishes. Since a closed curve can be regarded as the limiting case of a polygon, *expression* (28) *applies also to contours limited by a continuous curve.* Finally, it is easy to see that *the limiting value* (28) *will be correct also for three dimensional convex bodies.* Though our theory gives no quantitative description of the three dimensional case, qualitatively the phenomenon is the same in both cases. Especially it will be true, also in the three dimensional case, that in the limit of very high velocities the surface of discontinuity (shock wave) will closely envelop the surface of the body and the region of compression will form only a thin film on this surface. Since every small part of a thin film can be considered as plane, we can apply to it the formulas developed for the two dimensional case and arrive, in this way, at expression (28).

To avoid misunderstandings it should be emphasized that, in the preceding paragraphs, a velocity is considered "*high,*" if it is capable of causing a large compression x. The point, where low velocities end and high velocities begin, depends entirely on the compressibility of the medium.

[1] B. Riemann, Gesammelte Werke, p. 156. Also: Riemann-Weber, Partielle Differentialgleichungen, Vol. II., pp. 469–498. Braunschweig 1901.

[2] L. Prandtl, *Phys. Zs.*, **5**, p. 599 (1904).

[3] Th. Meyer, Mitteilungen des Vereins deutscher Ingenieure. Berlin 1908.

[4] E. Mach, *Wien. Ber.*, **77**, p. 7; p. 819 (1878).

[5] This reasoning is accurate only then when the velocity of sound is the same in all directions.

[6] For large front angles (30° and 45°) and low velocities (vicinity of w_{min}), w_2 comes out smaller than the velocity of sound a_2. This result may have no physical reality because of thermal conduction: compare section 9.

[7] C. Cranz, Äussere Ballistik. Vol. I. Berlin 1925.

[8] F. Klein and A. Sommerfeld, "Theorie des Kreisels." Bd. IV, Chapter "Ballistik." Leipzig, 1910.

[9] H. Lorenz, *Zs. Vereins deutscher Ingenieure*, p. 625 (1916).

[10] This argument is not rigorous, but it is possible to give a rigorous proof.

HIGH VACUUM PRESSURE CONTROL APPARATUS

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In an experiment such as the measurement of the speed of a diffusion pump, if one is interested in a particular pressure maintained for a long interval, a pressure control apparatus is needed. For maintaining a low vacuum pressure, Richards' apparatus[1] and Van den Akker's attachment to a U-manometer[2] are the satisfactory devices for automatic control. But they cease to function for a high vacuum. The present device could be used for a high vacuum as well as for a moderate vacuum.

The arrangement of the apparatus is shown in the diagram. A is a vacuum system in which a constant pressure is to be maintained, and R is a reservoir the pressure of which is different from that of A. The communication between A and R can be made by opening the electric mercury valve S, designed to open and close automatically.

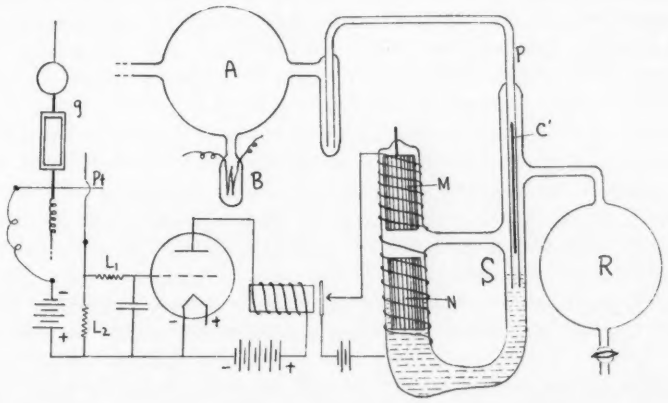
Since the pressure in A may be measured by a hot wire gauge (or an ionization gauge) B, this pressure is indicated by the deflection of the galvanometer. Then, the galvanometer, owing to its deflection, can be used as a contact key (called "galvanometer contact"). By the action of the "galvanometer contact" and a system of relays, it is easy to operate the electric mercury valve.

To form a "galvanometer contact," as shown in the diagram, g being the galvanometer coil and mirror (the author used a Leeds and Northrup Co. type "P" galvanometer), use two pieces of fine platinum wire Pt, one attached horizontally to the lower side of the galvanometer coil, the other fixed vertically on the side of the lower suspension, both should be carefully insulated from the galvanometer circuit, and their relative positions should be such that at certain required deflected-position of the mirror (i.e., at certain required pressure in A) they make contact with each other.

But a very small current in the "galvanometer contact" will cause two wires to stick together. This kind of difficulty occurs usually in small current relays as pointed out and eliminated by Roberts.[3] To eliminate this, let the "galvanometer contact" control a vacuum tube amplifier. When the two platinum wires are in contact, they produce a sufficient negative voltage on the grid to reduce the plate current nearly to zero, and thus to operate the relay in the plate circuit. For the amplifier, tube 201A, plate voltage = 120 volts, L_1 = 10,000 ohms, L_2 = 10 megohms and grid voltage = -20 volts were used. The following precaution should be taken: One should not apply too large a voltage to the plate circuit, because it will produce an ionization current in the grid circuit,

and hence in the "galvanometer contact," and so revive the difficulty of sticking.

The electric mercury valve is a U-tube half filled with mercury. An iron cylinder N, floating on the mercury surface in one limb of the U-tube, can be held and released by an electromagnet M. When N is released the mercury surface in the other limb will rise and close the mouth of the tube P; and when N is held the mercury will then open the mouth of P, thus forming a valve. Since the response of the gauge B to the change of the pressure is not instantaneous, the deflection of the galvanometer lags behind the actual pressure, if the latter is either quickly increasing or quickly decreasing. So, in case the difference of pressure between A and R is large it is necessary to reduce the rate of flow through the valve by inserting in P a capillary tube C′ of suitable dimensions. The U-tube



was made of 2 cm. tubing on the left side, and of 1 cm. tubing on the right side; P was of 0.6 cm. tubing; both M and N were 4 cm. long and 1.8 cm. in diameter.

Suppose that the pressure in the vacuum system A is gradually decreasing. Choose the proper direction of the galvanometer current. The "galvanometer contact" is then made. The action of the contact will open the mercury valve to increase the pressure. In this case R should have a higher pressure than A. When the pressure is restored, the "galvanometer contact" breaks and the valve is then closed again. If the pressure in A is gradually increasing, R should have a lower pressure than A.

The higher the sensitivity of the gauge B, the smaller would be the magnitude of the fluctuation of the pressure in A.

The above device is a part of a low pressure gas flowmeter[4] which the author designed to measure speeds of diffusion pumps.

For more accurate work, it is suggested that a selenium cell (or a photoelectric cell) be exposed to the image of a strongly illuminated source reflected at the galvanometer mirror, instead of using the platinum wire "galvanometer contact." The remaining part of the electric circuit is exactly the same. This is similar to Van den Akker's attachment used in this laboratory.

The author wishes to express his thanks to Professor A. Goetz who assisted in the preparation of this paper.

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[1] L. A. Richards, *R. S. I.*, **2**, p. 49, 1931.

[2] J. A. Van den Akker, *R. S. I.*, **1**, p. 684, 1930.

[3] Howard S. Roberts, *J. O. S. A.* & *R. S. I.*, **11**, p. 179, 1925.

[4] T. L. Ho, Dissertation, China Foundation, 1931.